

# ASYMPTOTIC MULTIVARIATE EXPECTILES

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ABSTRACT. In [?] , a new family of vector-valued risk measures called *multivariate expectiles* is introduced. In this paper, we focus on the asymptotic behavior of these measures in a multivariate regular variations context. For models with equivalent tails, we propose an estimator of these multivariate asymptotic expectiles, in the Fréchet attraction domain case, with asymptotic independence, or in the comonotonic case.

## INTRODUCTION

In few years, expectiles became an important risk measure among more used ones, essentially, because it satisfies both coherence and elicibility properties. In dimension one, expectiles are introduced by Newey and Powell (1987) [? ]. For a random variable  $X$  with finite order 2 moment, the expectile of level  $\alpha$  is defined as

$$e_\alpha(X) = \arg \min_{x \in \mathbb{R}} \mathbb{E}[\alpha(X - x)_+^2 + (1 - \alpha)(x - X)_+^2],$$

where  $(x)_+ = \max(x, 0)$ . Expectiles are the only risk measure satisfying both elicibility and coherence properties, according to Bellini and Bignozzi (2015) [? ].

In higher dimension, one of the proposed extensions of expectiles in [?] is the *Matrix Expectiles*. Consider a random vector  $\mathbf{X} = (X_1, \dots, X_d)^T \in \mathbb{R}^d$  having order 2 moments, and let  $\Sigma = (\pi_{ij})_{1 \leq i, j \leq d}$  be a  $d \times d$  real matrix, symmetric and positive semi-definite such that  $i \in \{1, \dots, d\}$ ,  $\pi_{ii} = \pi_i > 0$ .

A  $\Sigma$ -expectile of  $\mathbf{X}$ , is defined as

$$\mathbf{e}_\alpha^\Sigma(\mathbf{X}) \in \arg \min_{\mathbf{x} \in \mathbb{R}^d} \mathbb{E}[\alpha(\mathbf{X} - \mathbf{x})_+^T \Sigma (\mathbf{X} - \mathbf{x})_+ + (1 - \alpha)(\mathbf{X} - \mathbf{x})_-^T \Sigma (\mathbf{X} - \mathbf{x})_-],$$

where  $(\mathbf{x})_+ = ((x_1)_+, \dots, (x_d)_+)^T$  and  $(\mathbf{x})_- = (-\mathbf{x})_+$ . We shall concentrate on the case where the above minimization has a unique solution. In [? ], conditions on  $\Sigma$  ensuring the uniqueness of the argmin are given, it is sufficient that  $\pi_{ij} \geq 0$ ,  $\forall i, j \in \{1, \dots, d\}$ . We shall make this assumption throughout this paper. Then, the vector expectile is unique, and it is the solution of the following equation system

$$(0.1) \quad \alpha \sum_{i=1}^d \pi_{ki} \mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_k > x_k\}}] = (1 - \alpha) \sum_{i=1}^d \pi_{ki} \mathbb{E}[(x_i - X_i)_+ \mathbb{1}_{\{x_k > X_k\}}], \quad \forall k \in \{1, \dots, d\}.$$

When  $\pi_{ij} = 1$  for all  $(i, j) \in \{1, \dots, d\}^2$ , the corresponding  $\Sigma$ -expectile is called a  $L_1$ -expectile. It coincides with the  $L_1$ -norm expectile defined in [? ].

In [?] it is proved that,

$$\lim_{\alpha \rightarrow 1} \mathbf{e}_\alpha^\Sigma(\mathbf{X}) = \mathbf{X}_\mathbf{F}, \quad \text{and} \quad \lim_{\alpha \rightarrow 0} \mathbf{e}_\alpha^\Sigma(\mathbf{X}) = \mathbf{X}_\mathbf{I},$$

where  $\mathbf{X}_\mathbf{F} \in (\mathbb{R} \cup \{+\infty\})^d$  is the right endpoint vector  $(x_F^1, \dots, x_F^d)^T$ , and by  $\mathbf{X}_\mathbf{I} \in (\mathbb{R} \cup \{-\infty\})^d$  is the left endpoint vector  $(x_I^1, \dots, x_I^d)^T$  of the support of the random vector  $\mathbf{X}$ .

The asymptotic levels i.e.  $\alpha \rightarrow 1$  or  $\alpha \rightarrow 0$  represent extreme risks. For example, the solvency thresholds in insurance, are generally asymptotic, so that the asymptotic behavior of the risk measures is of natural importance.

The multivariate expectiles can be estimated in the general case using stochastic optimization algorithms. The example of estimation by the Robbins-Monro's (1951) [?] algorithm, presented in [? ], shows that in the asymptotic level, the obtained estimation is not satisfactory in term of convergence speed. This leads us to the theoretical analysis of the asymptotic behavior of multivariate expectiles.

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We shall work focus on the equivalent tails model. It is often used in modeling the claim amounts in insurance, in studying dependent extreme events, and in ruin theory models. This model includes in particular the identically distributed portfolios of risks and the case with scale difference in the distributions. In this paper, we study the asymptotic behavior of multivariate expectiles in the multivariate regular variations framework. We focus on marginal distributions belonging to Fréchet domain of attraction. This domain contains the heavy-tailed distributions that represent the most dangerous claims in insurance. Let us remark that the attention to univariate expectiles is recent. In [? ], asymptotic equivalents of expectiles as a function of the quantile of the same level for regular variation distributions are proved. First and second order asymptotics for the expectile of the sum in the case of FGM dependence structure are given in [? ].

The paper is constructed as follows. The first section is devoted to the presentation of the multivariate regularly varying distribution framework. The study of the asymptotic behavior of the multivariate expectiles for Fréchet model with equivalent tails is the subject of Section 2. The case of an asymptotically dominant tail is analyzed in Section 3. Section 4 is a presentation of some estimators of the asymptotic expectiles in the cases of asymptotic independence and comonotonicity.

## 1. THE MRV FRAMEWORK

Regularly varying distributions are well suited to study extreme phenomenons. Lots of works have been devoted to the asymptotic behavior of usual risk measures for this class of distributions, and results are given for sums of risks belonging to this family. It is well known that the three domains of attraction of extreme value distributions can be defined using the concept of regular variations (see [? ? ? ? ]).

This section is devoted to the classical characterization of multivariate regular variations, which will be used in the study of the asymptotic behavior of multivariate expectiles. We also recall some basic results on the univariate setting that we shall use.

**1.1. Univariate regular variations.** We begin by recalling basic definitions and results on univariate regular variations.

**Definition 1.1** (Regularly varying functions). A measurable positive function  $f$  is *regularly varying of index  $\rho$  at  $a \in \{0, +\infty\}$* , if for all  $t > 0$ ,

$$\lim_{x \rightarrow a} \frac{f(tx)}{f(x)} = t^\rho,$$

we denote  $f \in \text{RV}_\rho(a)$ .

A slowly varying function is a regularly varying function of index  $\rho = 0$ . Remark that a function  $f \in \text{RV}_\rho(+\infty)$  if and only if, there exists a slowly varying function at infinity,  $L \in \text{RV}_0(+\infty)$  such that

$$f(x) = x^\rho L(x).$$

**Theorem 1.2** (Karamata's representation, [? ]). *For any slowly varying function  $L$  at  $+\infty$ , there exist a positive measurable function  $c(\cdot)$  that satisfies  $\lim_{x \rightarrow +\infty} c(x) = c \in ]0, +\infty[$ , and a measurable function  $\varepsilon(\cdot)$  with  $\lim_{x \rightarrow +\infty} \varepsilon(x) = 0$ , such that*

$$L(x) = c(x) \exp \left( \int_1^x \frac{\varepsilon(t)}{t} dt \right).$$

The Karamata's representation is generalized to RV functions. Indeed,  $f \in \text{RV}_\rho(+\infty)$  if and only if it can written in the form

$$f(x) = c(x) \int_1^x \frac{\rho(t)}{t} dt,$$

where  $\rho(t) = \rho$  and  $c(t) = c \in ]0, +\infty[$ . Throughout the paper, we shall consider generalized inverses of non-decreasing functions  $f$ :  $f^{\leftarrow}(y) = \inf\{x \in \mathbb{R}, f(x) \geq y\}$ .

**Lemma 1.3** (Inverse of RV functions [? ]). *Let  $f \in \text{RV}_\rho(a)$  be non decreasing. If  $\rho > 0$  then  $f^{\leftarrow} \in \text{RV}_{1/\rho}(f(a))$  and if  $\rho < 0$  then  $f^{\leftarrow} \in \text{RV}_{-1/\rho}(-f(a))$ .*

In [? ], it is proven that for a measurable non negative function  $f$ , defined on  $\mathbb{R}^+$ , such that  $\lim_{x \rightarrow +\infty} f(x) = +\infty$

$$f \in \text{RV}_\rho(+\infty) \text{ if and only if } f^{\leftarrow} \in \text{RV}_{\frac{1}{\rho}}(+\infty).$$

**Lemma 1.4** (Integration of RV functions (Karamata's Theorem)), [? ]. For a positive measurable function  $f$ , regularly varying of index  $\rho$  at  $+\infty$ , locally bounded on  $[x_0, +\infty)$  with  $x_0 \geq 0$

- if  $\rho > -1$ , then

$$\lim_{x \rightarrow +\infty} \frac{\int_{x_0}^x f(t) dt}{x f(x)} = \frac{1}{\rho + 1},$$

- if  $\rho < -1$ , then

$$\lim_{x \rightarrow +\infty} \frac{\int_x^{+\infty} f(t) dt}{x f(x)} = -\frac{1}{\rho + 1}.$$

**Lemma 1.5** (Potter's bounds [? ]). For  $f \in RV_\rho(a)$ , with  $a \in \{0, \infty\}$  et  $\rho \in \mathbb{R}$ . For any  $0 < \epsilon < 1$  and all  $x$  and  $y$  sufficiently close to  $a$ , we have

$$(1 - \epsilon) \min \left( \left( \frac{x}{y} \right)^{\rho - \epsilon}, \left( \frac{x}{y} \right)^{\rho + \epsilon} \right) \leq \frac{f(x)}{f(y)} \leq (1 + \epsilon) \max \left( \left( \frac{x}{y} \right)^{\rho - \epsilon}, \left( \frac{x}{y} \right)^{\rho + \epsilon} \right).$$

Many other properties of regularly varying functions are presented e.g. in [? ].

**1.2. Multivariate regular variations.** The multivariate extension of regular variations is introduced in literature in [? ]. We denote by  $\mu_n \xrightarrow{v} \mu$  the vague convergence of Radon measures as presented in [? ]. The following definitions are given for non negative random variables.

**Definition 1.6** (Multivariate regular variations). The distribution of a random vector  $\mathbf{X}$  on  $[0, \infty]^d$  is said to regularly varying if there exist a non-null Radon measure  $\mu_{\mathbf{X}}$  on the Borel  $\sigma$ -algebra  $\mathcal{B}_d$  on  $[0, \infty]^d \setminus \mathbf{0}$ , and a normalization function  $b : \mathbb{R} \rightarrow \mathbb{R}$  which satisfies  $\lim_{x \rightarrow +\infty} b(x) = +\infty$  such that

$$(1.1) \quad u \mathbb{P} \left( \frac{\mathbf{X}}{b(u)} \in \cdot \right) \xrightarrow{v} \mu_{\mathbf{X}}(\cdot) \text{ as } u \rightarrow +\infty.$$

There exist several equivalent definitions of multivariate regular variations which will be useful in what follows.

**Definition 1.7** (MRV equivalent Definitions). Let  $\mathbf{X}$  be a random vector on  $\mathbb{R}^d$ , the following definitions are equivalent:

- The vector  $\mathbf{X}$  has a regularly varying tail of index  $\theta$ .
- There exist a finite measure  $\mu$  on the unit sphere  $\mathbb{S}^{d-1}$ , and a normalization function  $b : (0, \infty) \rightarrow (0, \infty)$  such that

$$(1.2) \quad \lim_{t \rightarrow +\infty} \mathbb{P} \left( \|\mathbf{X}\| > x b(t), \frac{\mathbf{X}}{\|\mathbf{X}\|} \in \cdot \right) = x^{-\theta} \mu(\cdot),$$

for all  $x > 0$ . The measure  $\mu$  depends on the chosen norm, it is called the *spectral measure* of  $\mathbf{X}$ .

- There exists a finite measure  $\mu$  on the unit sphere  $\mathbb{S}^{d-1}$ , a slowly varying function  $L$ , and a positive real  $\theta > 0$  such that

$$(1.3) \quad \lim_{x \rightarrow +\infty} \frac{x^\theta}{L(x)} \mathbb{P} \left( \|\mathbf{X}\| > x, \frac{\mathbf{X}}{\|\mathbf{X}\|} \in B \right) = \mu(B),$$

for all  $B \in \mathcal{B}(\mathbb{S}^{d-1})$  with  $\mu(\partial B) = 0$ .

From now on, MRV denotes the set of multivariate regularly varying distributions, and by  $\text{MRV}(\theta, \mu)$  denotes the set of random vectors with regularly varying tail, with index  $\theta$  and spectral measure  $\mu$ .

From (1.3), we may assume that  $\mu$  is normalized i.e.  $\mu(\mathbb{S}^{d-1}) = 1$ , which implies that  $\|\mathbf{X}\|$  has a regularly varying tail of index  $-\theta$ .

On another hand,

$$\begin{aligned} \lim_{x \rightarrow +\infty} \mathbb{P} \left( \frac{\mathbf{X}}{\|\mathbf{X}\|} \in B \mid \|\mathbf{X}\| > x \right) &= \lim_{x \rightarrow +\infty} \frac{\mathbb{P} \left( \|\mathbf{X}\| > x, \frac{\mathbf{X}}{\|\mathbf{X}\|} \in B \right)}{\mathbb{P}(\|\mathbf{X}\| > x)} \\ &= \lim_{x \rightarrow +\infty} \frac{x^\theta}{L(x)} \mu(B) x^{-\theta} L(x) = \mu(B), \end{aligned}$$

for all  $B \in \mathcal{B}(\mathbb{S}^{d-1})$  with  $\mu(\partial B) = 0$ . That means that conditionally to  $\{\|\mathbf{X}\| > x\}$ ,  $\frac{\mathbf{X}}{\|\mathbf{X}\|}$  converges weakly to  $\mu$ .

The different possible characterizations of the MRV concept are presented in [? ].

**1.3. Characterization using tail dependence functions.** Let  $\mathbf{X} = (X_1, \dots, X_d)$  be a random vector. From now on,  $\bar{F}_{X_i}$  denotes the survival function of  $X_i$ . We assume that  $\mathbf{X}$  has equivalent regularly varying marginal tails, which means:

**H1:**  $\bar{F}_{X_1} \in \text{RV}_{-\theta}(+\infty)$ , with  $\theta > 0$ .

**H2:** The tails of  $X_i, i = 1, \dots, d$  are equivalent. That is for all  $i \in \{2, \dots, d\}$ , there is a positive constant  $c_i$  such that

$$\lim_{x \rightarrow +\infty} \frac{\bar{F}_{X_i}(x)}{\bar{F}_{X_1}(x)} = c_i.$$

H1 and H2 imply that all marginal tails are regularly varying of index  $-\theta$  at  $+\infty$ .

In this paper, we use the definition of the upper tail dependence function, as introduced in [? ].

**Definition 1.8** (The tail dependence function). Let  $\mathbf{X}$  be a random vector on  $\mathbb{R}^d$ , with continuous marginal distributions. The tail dependence function is defined by

$$(1.4) \quad \lambda_U^{\mathbf{X}}(x_1, \dots, x_d) = \lim_{t \rightarrow 0^+} t^{-1} \mathbb{P}(\bar{F}_{X_1}(X_1) \leq tx_1, \dots, \bar{F}_{X_d}(X_d) \leq tx_d),$$

when the limit exists.

For  $k \leq d$ , denote by  $X^{(k)}$  a  $k$  dimensional sub-vector of  $\mathbf{X}$ ,  $C^{(k)}$  its copula and  $\bar{C}^{(k)}$  its survival copula. The upper tail dependence function is

$$(1.5) \quad \lambda_U^k(u_1, \dots, u_k) = \lim_{t \rightarrow 0^+} \frac{\bar{C}_k(tu_1, \dots, tu_k)}{t},$$

if this limit exists. The lower tail dependence function can be defined analogically by

$$\lambda_L^k(u_1, \dots, u_k) = \lim_{t \rightarrow 0^+} \frac{C_k(tu_1, \dots, tu_k)}{t},$$

when the limit exists. In this paper, our study is limited to the upper version as defined in (1.5).

The following two theorems show that, under H1 and H2, the MRV character of multivariate distributions is equivalent to the existence of the tail dependence functions.

**Theorem 1.9** (Theorem 2.3 in [? ]). *Let  $\mathbf{X} = (X_1, \dots, X_d)$  be a random vector in  $\mathbb{R}^d$ , with continuous marginal distributions  $F_{X_i}, i = 1, \dots, d$  that satisfy H1 and H2. If  $\mathbf{X}$  has an MRV distribution, the tail dependence function exists, and it is given by par*

$$\lambda_U^k(u_1, \dots, u_k) = \lim_{x \rightarrow +\infty} x \mathbb{P} \left( X_1 > b(x) \left( \frac{u_1}{c_1} \right)^{-1/\theta}, \dots, X_k > b(x) \left( \frac{u_d}{c_d} \right)^{-1/\theta} \right),$$

for any  $k \in \{1, \dots, d\}$ .

**Theorem 1.10** (Theorem 3.2 in [? ]). *Let  $\mathbf{X} = (X_1, \dots, X_d)$  be a random vector in  $\mathbb{R}^d$ , with continuous marginal distributions  $F_{X_i}, i = 1, \dots, d$  that satisfies H1 and H2. If the tail dependence function  $\lambda_U^k$  exists for all  $k \in \{1, \dots, d\}$ , then  $\mathbf{X}$  is MRV, its normalization function is given by  $b(u) = \left( \frac{1}{F_{X_1}} \right)^{\leftarrow} (u)$  and the spectral measure is*

$$\mu([0, \mathbf{x}]^c) = \sum_{i=1}^d c_i x_i^{-\theta} - \sum_{1 \leq i < j \leq d} \lambda_U^2(c_i x_i^{-\theta}, c_j x_j^{-\theta}) + \dots + (-1)^{d+1} \lambda_U^d(c_1 x_1^{-\theta}, \dots, c_d x_d^{-\theta}).$$

By construction of the multivariate expectiles, only the bivariate dependence structures are taken into account. We shall use the functions  $\lambda_U^{(X_i, X_k)}$ , for all  $(i, k) \in \{1, \dots, d\}^2$ . In order to simplify the notation, we denote it by  $\lambda_U^{ik}$ . If the vector  $\mathbf{X}$  has an MRV distribution, the pairs  $(X_i, X_j)$  have also MRV distributions, for any  $(i, j) \in \{1, \dots, d\}^2$ . So, in the MRV framework, and under H1 and H2, the existence of functions  $\lambda^{ik}$  is insured. In addition, we assume in all the rest of this paper that these functions are continuous.

## 2. FRÉCHET MODEL WITH EQUIVALENT TAILS

In this section, we assume that  $\mathbf{X}$  satisfies H1 and H2 with  $\theta > 1$ . It implies that  $X_1$  belongs to the extreme value domain of attraction of Fréchet MDA( $\Phi_\theta$ ). This domain contains distributions with infinite endpoint  $x_F = \sup\{x : F(x) < 1\} = +\infty$ , so as  $\alpha \rightarrow 1$  we get  $e_\alpha^i(\mathbf{X}) \rightarrow +\infty \forall i$ . Also, from Karamata's Theorem (Theorem 1.4), we have for  $i = 1, \dots, d$ ,

$$(2.1) \quad \lim_{x \rightarrow +\infty} \frac{\mathbb{E}[(X_i - x)_+]}{x \bar{F}_{X_i}(x)} = \frac{1}{\theta - 1},$$

for all  $i \in \{1, \dots, d\}$ .

**Proposition 2.1.** *Let  $\Sigma = (\pi_{i,j})_{i,j=1,\dots,d}$  with  $\pi_{ij} > 0$  for all  $i, j \in \{1, \dots, d\}$ . Under H1 and H2, the components of the multivariate  $\Sigma$ -expectiles  $\mathbf{e}_\alpha(\mathbf{X}) = (e_\alpha^i(\mathbf{X}))_{i=1,\dots,d}$  satisfy*

$$0 < \lim_{\alpha \rightarrow 1} \frac{e_\alpha^i(\mathbf{X})}{e_\alpha^1(\mathbf{X})} \leq \lim_{\alpha \rightarrow 1} \frac{e_\alpha^i(\mathbf{X})}{e_\alpha^1(\mathbf{X})} < +\infty, \forall i \in \{2, \dots, d\}.$$

Proposition 2.1 implies that distributions with equivalent tails have asymptotically comparable multivariate expectile components.

Before we prove Proposition 2.1, we shall demonstrate some preliminary results. Firstly, let  $\mathbf{X} = (X_1, \dots, X_d)^T$  satisfy H1 and H2, we denote  $x_i = e_\alpha^i(\mathbf{X})$  for all  $i \in \{1, \dots, d\}$ . We define the functions  $l_{X_i, X_j}^\alpha$  for all  $(i, j) \in \{1, \dots, d\}^2$  by

$$(2.2) \quad l_{X_i, X_j}^\alpha(x_i, x_j) = \alpha \mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_j > x_j\}}] - (1 - \alpha) \mathbb{E}[(X_i - x_i)_- \mathbb{1}_{\{X_j < x_j\}}],$$

and  $l_{X_i}^\alpha(x_i) = l_{X_i, X_i}^\alpha(x_i, x_i)$ .

The optimality system (0.1) rewrites

$$(2.3) \quad l_{X_k}^\alpha(x_k) = - \sum_{i=1, i \neq k}^d \frac{\pi_{ki}}{\pi_{kk}} l_{X_i, X_k}^\alpha(x_i, x_k) \quad \forall k \in \{1, \dots, d\}.$$

We shall use the following sets:

$$\begin{aligned} J_0^i &= \{j \in \{1, \dots, d\} \setminus \{i\} \mid \lim_{\alpha \rightarrow 1} \frac{x_j}{x_i} = 0\}, \\ J_C^i &= \{j \in \{1, \dots, d\} \setminus \{i\} \mid 0 < \lim_{\alpha \rightarrow 1} \frac{x_j}{x_i} < \lim_{\alpha \rightarrow 1} \frac{x_j}{x_i} < +\infty\}, \\ \text{and } J_\infty^i &= \{j \in \{1, \dots, d\} \setminus \{i\} \mid \lim_{\alpha \rightarrow 1} \frac{x_j}{x_i} = +\infty\}. \end{aligned}$$

The proof of Proposition 2.1 is written for  $\pi_{ij} = 1$ , for all  $(i, j) \in \{1, \dots, d\}^2$ , ie for the  $L_1$ -expectiles. The general case can be treated in the same way, provided that  $\pi_{ij} > 0$  for all  $(i, j) \in \{1, \dots, d\}^2$ . The proof of Proposition 2.1 follows from Lemma 2.2 and Propositions 2.3 and 2.4 below.

**Lemma 2.2.** *Assume that H1 and H2 are satisfied.*

(1) *If  $t = o(s)$  then for all  $(i, j) \in \{1, \dots, d\}^2$ ,*

$$\lim_{t \uparrow +\infty} \frac{s \bar{F}_{X_i}(s)}{t \bar{F}_{X_j}(t)} = 0.$$

(2) *If  $t = \Theta(s)$ ,<sup>1</sup> then for all  $(i, j) \in \{1, \dots, d\}^2$ ,*

$$\frac{\bar{F}_{X_i}(s)}{\bar{F}_{X_j}(t)} \sim \frac{c_i}{c_j} \left(\frac{s}{t}\right)^{-\theta} \quad \text{as } t \rightarrow \infty.$$

*Proof.* We give some details on the proof for the first item, the second one may be obtained in the same way. Under H1 and H2, for all  $i \in \{1, \dots, d\}$

$$\bar{F}_{X_i} \in \text{RV}_{-\theta}(+\infty),$$

there exists for all  $i$  a positive measurable function  $L_i \in \text{RV}_0(+\infty)$  such that

$$\bar{F}_{X_i}(x) = x^{-\theta} L_i(x), \forall x > 0,$$

then for all  $(i, j) \in \{1, \dots, 2\}^2$  and all  $t, s > 0$

$$(2.4) \quad \frac{s \bar{F}_{X_i}(s)}{t \bar{F}_{X_j}(t)} = \left(\frac{s}{t}\right)^{-\theta+1} \frac{L_i(s)}{L_j(t)} = \left(\frac{s}{t}\right)^{-\theta+1} \frac{L_i(s)}{L_i(t)} \frac{L_i(t)}{L_j(t)},$$

and under H2

$$(2.5) \quad \lim_{x \uparrow +\infty} \frac{L_i(x)}{L_j(x)} = \frac{c_i}{c_j}.$$

Using Karamata's representation for slowly varying functions (Theorem 1.2), there exist a constant  $c > 0$ , a positive measurable function  $c(\cdot)$  with  $\lim_{x \uparrow +\infty} c(x) = c > 0$ , such that  $\forall \epsilon > 0, \exists t_0$  such that  $\forall t > t_0$

<sup>1</sup>Recall that  $t = \Theta(s)$  means that there exist positive constants  $C_1$  and  $C_2$  such that  $C_1 s \leq t \leq C_2 s$

$$\frac{L_i(s)}{L_i(t)} \leq \left(\frac{s}{t}\right)^\epsilon \frac{c(s)}{c(t)}.$$

Taking  $0 < \epsilon < \theta - 1$ , we conclude

$$\lim_{t \uparrow +\infty} \frac{s \bar{F}_{X_i}(s)}{t \bar{F}_{X_j}(t)} = 0, \quad \forall (i, j) \in \{1, \dots, d\}^2.$$

□

**Proposition 2.3.** *Under H1 and H2, the components of the asymptotic multivariate expectile satisfy*

$$\overline{\lim}_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_i}(\mathbf{e}_\alpha^i(\mathbf{X}))} < +\infty, \quad \forall i \in \{2, \dots, d\}.$$

*Proof.* Using H2, it is sufficient to show that

$$\overline{\lim}_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))} < +\infty.$$

Assume that  $\overline{\lim}_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))} = +\infty$ , we shall prove that, in that case, (2.3) cannot be satisfied. Taking if necessary a subsequence  $(\alpha_n \rightarrow 1)$ , we may assume that  $\lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))} = +\infty$ .

We have

$$\begin{aligned} \frac{l_{X_1}^\alpha(x_1)}{(1 - \alpha)x_1} &= \left( \frac{\alpha \mathbb{E}[(X_1 - x_1)_+] - (1 - \alpha) \mathbb{E}[(X_1 - x_1)_-]}{(1 - \alpha)x_1} \right) \\ &= \left( (2\alpha - 1) \frac{\mathbb{E}[(X_1 - x_1)_+]}{(1 - \alpha)x_1} - \frac{(1 - \alpha)(x_1 - \mathbb{E}[X_1])}{(1 - \alpha)x_1} \right) \\ &= \left( (2\alpha - 1) \frac{\mathbb{E}[(X_1 - x_1)_+]}{x_1 \bar{F}_{X_1}(x_1)} \frac{\bar{F}_{X_1}(x_1)}{1 - \alpha} - 1 + \frac{\mathbb{E}[X_1]}{x} \right) \xrightarrow{\alpha \uparrow 1} -1 \text{ recall (2.1)}. \end{aligned}$$

Furthermore, for all  $i \in \{2, \dots, d\}$

$$\begin{aligned} \frac{l_{X_i, X_1}^\alpha(x_i, x_1)}{(1 - \alpha)x_1} &= \left( \frac{\alpha \mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}] - (1 - \alpha) \mathbb{E}[(X_i - x_i)_- \mathbb{1}_{\{X_1 < x_1\}}]}{(1 - \alpha)x_1} \right) \\ &= \left( \frac{\alpha \mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{(1 - \alpha)x_1} - \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 < x_1\}}]}{x_1} - \frac{x_i \mathbb{P}(X_1 < x_1)}{x_1} + \frac{\mathbb{E}[X_i \mathbb{1}_{\{X_1 < x_1\}}]}{x_1} \right). \end{aligned}$$

On one side,

$$\frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{(1 - \alpha)x_1} \leq \frac{\mathbb{E}[(X_i - x_i)_+]}{(1 - \alpha)x_1} = \frac{\bar{F}_{X_1}(x_1)}{1 - \alpha} \frac{\mathbb{E}[(X_i - x_i)_+]}{x_i \bar{F}_{X_i}(x_i)} \frac{x_i \bar{F}_{X_i}(x_i)}{x_1 \bar{F}_{X_1}(x_1)}, \quad \forall i \in \{2, \dots, d\}.$$

So that, Lemma 2.2 implies

$$(2.6) \quad \lim_{\alpha \rightarrow 1} \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{(1 - \alpha)x_1} = 0, \quad \forall k \in J_C^1 \cup J_\infty^1,$$

Let  $i \in J_0^1$ , taking if necessary a subsequence, we may assume that  $\frac{x_i}{x_1} \rightarrow 0$ .

$$(2.7) \quad \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{(1 - \alpha)x_1} = \frac{\int_{x_i}^{x_1} \mathbb{P}(X_i > t, X_1 > x_1) dt}{(1 - \alpha)x_1} + \frac{\int_{x_1}^{+\infty} \mathbb{P}(X_i > t, X_1 > x_1) dt}{(1 - \alpha)x_1}.$$

Now,

$$\frac{\int_{x_i}^{x_1} \mathbb{P}(X_i > t, X_1 > x_1) dt}{(1 - \alpha)x_1} \leq \frac{\int_{x_i}^{x_1} \mathbb{P}(X_1 > x_1) dt}{(1 - \alpha)x_1} = \frac{\bar{F}_{X_1}(x_1)}{1 - \alpha} \left( 1 - \frac{x_i}{x_1} \right).$$

Thus,

$$(2.8) \quad \lim_{\alpha \rightarrow 1} \frac{\int_{x_i}^{x_1} \mathbb{P}(X_i > t, X_1 > x_1) dt}{(1 - \alpha)x_1} = 0.$$

Consider the second term of (2.7)

$$\frac{\int_{x_1}^{+\infty} \mathbb{P}(X_i > t, X_1 > x_1) dt}{(1 - \alpha)x_1} \leq \frac{\int_{x_1}^{+\infty} \mathbb{P}(X_i > t) dt}{(1 - \alpha)x_1},$$

Karamata's Theorem (Theorem 1.4) gives

$$\frac{\int_{x_1}^{+\infty} \mathbb{P}(X_k > t) dt}{(1 - \alpha)x_1} \underset{\alpha \uparrow 1}{\sim} \frac{1}{\theta - 1} \frac{\bar{F}_{X_k}(x_1)}{1 - \alpha},$$

which leads to

$$(2.9) \quad \lim_{\alpha \rightarrow 1} \frac{\int_{x_1}^{+\infty} \mathbb{P}(X_i > t, X_1 > x_1) dt}{(1 - \alpha)x_1} = 0.$$

Finally, we get

$$(2.10) \quad \lim_{\alpha \rightarrow 1} \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{(1 - \alpha)x_1} = 0, \forall i \in J_0^1.$$

We have shown that

$$\lim_{\alpha \rightarrow 1} \frac{\mathbb{E}[(X_k - x_k)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{(1 - \alpha)x_1} = 0, \forall k \in \{2, \dots, d\},$$

so, the first equation of optimality system (2.3) implies that

$$-\lim_{\alpha \rightarrow 1} \left( \sum_{k \in J_0^1 \setminus J_\infty^1} \frac{l_{X_k, X_1}^\alpha(x_k, x_1)}{(1 - \alpha)x_1} + \sum_{k \in J_C^1} \frac{l_{X_k, X_1}^\alpha(x_k, x_1)}{(1 - \alpha)x_1} + \sum_{k \in J_\infty^1} \frac{l_{X_k, X_1}^\alpha(x_k, x_1)}{(1 - \alpha)x_1} \right) = \lim_{\alpha \rightarrow 1} \sum_{k=2}^d \frac{x_k}{x_1} = -1,$$

this is absurd since the  $x_k$ 's are non negative, and consequently

$$\lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} < +\infty.$$

□

**Proposition 2.4.** *Under H1 and H2, the components of the asymptotic multivariate expectile satisfy*

$$0 < \lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_i}(\mathbf{e}_\alpha^i(\mathbf{X}))}, \forall i \in \{2, \dots, d\}.$$

*Proof.* Using H2, it is sufficient to show that

$$0 < \lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))}.$$

Let us assume that  $\lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))} = 0$ , we shall see that in that case, (2.3) cannot be satisfied. Taking if necessary a convergent subsequence, we may assume that  $\lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))} = 0$ . In this case,

$$\frac{l_{X_1}^\alpha(x_1)}{x_1 \bar{F}_{X_1}(x_1)} = \left( (2\alpha - 1) \frac{\mathbb{E}[(X_1 - x_1)_+]}{x_1 \bar{F}_{X_1}(x_1)} - \frac{1 - \alpha}{\bar{F}_1(x_1)} \left( 1 - \frac{\mathbb{E}[X_1]}{x_1} \right) \right) \xrightarrow{\alpha \uparrow 1} \frac{1}{\theta - 1} > 0.$$

On another side, let  $i \in J_\infty^1$ , taking if necessary a subsequence, we may assume that  $x_1 = o(x_i)$ . Lemma 2.2 and Proposition 2.3 give:

$$\frac{1 - \alpha}{\bar{F}_1(x_1)} \frac{x_i}{x_1} = \frac{1 - \alpha}{\bar{F}_{X_i}(x_i)} \cdot \frac{x_i \bar{F}_{X_i}(x_i)}{x_1 \bar{F}_{X_1}(x_1)} \rightarrow 0 \text{ as } \alpha \rightarrow 1.$$

Moreover,

$$\frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} \leq \frac{\mathbb{E}[(X_i - x_i)_+]}{x_1 \bar{F}_{X_1}(x_1)} = \frac{\mathbb{E}[(X_i - x_i)_+]}{x_i \bar{F}_{X_i}(x_i)} \frac{x_i \bar{F}_{X_i}(x_i)}{x_1 \bar{F}_{X_1}(x_1)},$$

We deduce

$$\begin{aligned} \frac{l_{X_i, X_1}^\alpha(x_i, x_1)}{x_1 \bar{F}_{X_1}(x_1)} &= \left( \frac{\alpha \mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} - (1 - \alpha) \frac{\mathbb{E}[(X_i - x_i)_- \mathbb{1}_{\{X_1 < x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} \right) \\ &\rightarrow 0, \forall i \in J_\infty^1. \end{aligned}$$

Going through the limit ( $\alpha \rightarrow 1$ ) in the first equation of the optimality system (2.3) divided by  $x_1 \bar{F}_{X_1}(x_1)$ , leads to

$$\lim_{\alpha \rightarrow 1} \sum_{k \in J_0^1 \cup J_C^1 \setminus J_\infty^1} \frac{l_{X_k, X_1}^\alpha(x_k, x_1)}{x_1 \bar{F}_{X_1}(x_1)} = -\frac{1}{\theta - 1},$$

which is absurd because

$$\begin{aligned} \lim_{\alpha \rightarrow 1} \sum_{k \in J_0^1 \cup J_C^1 \setminus J_\infty^1} \frac{l_{X_k, X_1}^\alpha(x_k, x_1)}{x_1 \bar{F}_{X_1}(x_1)} &= \lim_{\alpha \rightarrow 1} \sum_{k \in J_0^1 \cup J_C^1 \setminus J_\infty^1} \left( \frac{\alpha \mathbb{E}[(X_k - x_k)_+ \mathbb{1}_{\{X_1 > x_1\}}] - (1 - \alpha) \mathbb{E}[(X_k - x_k)_- \mathbb{1}_{\{X_1 < x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} \right) \\ &= \lim_{\alpha \rightarrow 1} \sum_{k \in J_0^1 \cup J_C^1 \setminus J_\infty^1} \left( \frac{\mathbb{E}[(X_k - x_k)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} - \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} \frac{x_k}{x_1} \right) \\ &= \lim_{\alpha \rightarrow 1} \sum_{k \in J_0^1 \cup J_C^1 \setminus J_\infty^1} \left( \frac{\mathbb{E}[(X_k - x_k)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} \right) \geq 0. \end{aligned}$$

We can finally conclude that

$$\lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} > 0.$$

□

The combination of Propositions 2.3 and 2.4 gives

$$0 < \lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_i}(\mathbf{e}_\alpha^i(\mathbf{X}))} \leq \overline{\lim}_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_i}(\mathbf{e}_\alpha^i(\mathbf{X}))} < +\infty, \forall i \in \{2, \dots, d\}.$$

We may now prove Proposition 2.1.

*Proof of Proposition 2.1.* We shall prove that  $J_\infty^1 = \emptyset$ , the fact that  $J_\infty^k = \emptyset$  for all  $k \in \{1, \dots, d\}$  may be proven in the same way. This implies that  $J_0^k = J_\infty^k = \emptyset$  for all  $k \in \{1, \dots, d\}$ , hence the result.

We suppose that  $J_\infty^1 \neq \emptyset$ , let  $i \in J_\infty^1$ , taking if necessary a subsequence, we may assume that  $x_i/x_1 \rightarrow +\infty$  as  $\alpha \rightarrow 1$ .

From Propositions 2.3 and 2.4, we have

$$0 < \lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_i}(\mathbf{e}_\alpha^i(\mathbf{X}))} \leq \overline{\lim}_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_i}(\mathbf{e}_\alpha^i(\mathbf{X}))} < +\infty, \forall i \in \{2, \dots, d\},$$

so, taking if necessary a subsequence, we may assume that  $\exists \ell \in \mathbb{R}^* \setminus \{+\infty\}$  such that

$$\lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} = \ell.$$

In this case,

$$\lim_{\alpha \rightarrow 1} \frac{l_{X_1}^\alpha(x_1)}{x_1 \bar{F}_{X_1}(x_1)} = \lim_{\alpha \rightarrow 1} \left( (2\alpha - 1) \frac{\mathbb{E}[(X_1 - x_1)_+]}{x_1 \bar{F}_{X_1}(x_1)} - \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} \left( 1 - \frac{\mathbb{E}[X_1]}{x_1} \right) \right) = \frac{1}{\theta - 1} - \ell < +\infty.$$

Moreover,

$$\frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} \leq \frac{\mathbb{E}[(X_i - x_i)_+]}{x_1 \bar{F}_{X_1}(x_1)} = \frac{\mathbb{E}[(X_i - x_i)_+]}{x_i \bar{F}_{X_i}(x_i)} \frac{x_i \bar{F}_{X_i}(x_i)}{x_1 \bar{F}_{X_1}(x_1)} \rightarrow 0 \text{ using Lemme 2.2.}$$

We get

$$\begin{aligned} \lim_{\alpha \rightarrow 1} \frac{l_{X_i, X_1}^\alpha(x_i, x_1)}{x_1 \bar{F}_{X_1}(x_1)} &= \lim_{\alpha \rightarrow 1} \left( \frac{\alpha \mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}] - (1 - \alpha) \mathbb{E}[(X_i - x_i)_- \mathbb{1}_{\{X_1 < x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} \right) \\ &= \lim_{\alpha \rightarrow 1} \left( \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} - \frac{1 - \alpha}{\bar{F}_{X_1}(x_1)} \frac{x_i}{x_1} \right) = -\infty, \forall i \in J_\infty^1. \end{aligned}$$

Going through the limit ( $\alpha \rightarrow 1$ ) in the first equation of the optimality System (2.3) divided by  $x_1 \bar{F}_{X_1}(x_1)$ , leads to

$$(2.11) \quad \lim_{\alpha \rightarrow 1} \sum_{k \in J_0^1 \cup J_C^1 \setminus J_\infty^1} \frac{l_{X_k, X_1}^\alpha(x_k, x_1)}{x_1 \bar{F}_{X_1}(x_1)} = -\infty.$$



Now, let  $k \in J_0^1$

$$\begin{aligned} \frac{\mathbb{E}[(X_k - x_k)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} &= \frac{\int_{x_k}^{x_1} \mathbb{P}(X_k > t, X_1 > x_1) dt}{x_1 \bar{F}_{X_1}(x_1)} + \frac{\int_{x_1}^{+\infty} \mathbb{P}(X_k > t, X_1 > x_1) dt}{x_1 \bar{F}_{X_1}(x_1)} \\ &\leq \frac{\int_{x_k}^{x_1} \mathbb{P}(X_1 > x_1) dt}{x_1 \bar{F}_{X_1}(x_1)} + \frac{\int_{x_1}^{+\infty} \mathbb{P}(X_k > t) dt}{x_1 \bar{F}_{X_1}(x_1)}, \end{aligned}$$

Karamata's Theorem (Theorem 1.4) leads to

$$\lim_{\alpha \rightarrow 1} \frac{\int_{x_k}^{x_1} \mathbb{P}(X_1 > x_1) dt}{x_1 \bar{F}_{X_1}(x_1)} + \frac{\int_{x_1}^{+\infty} \mathbb{P}(X_k > t) dt}{x_1 \bar{F}_{X_1}(x_1)} = 1 + \frac{c_k}{\theta - 1}, \forall k \in J_0^1.$$

Consider  $k \in J_C^1$

$$\frac{\mathbb{E}[(X_k - x_k)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} \leq \frac{\mathbb{E}[(X_k - x_k)_+]}{x_1 \bar{F}_{X_1}(x_1)} = \frac{\mathbb{E}[(X_k - x_k)_+]}{x_k \bar{F}_{X_k}(x_k)} \frac{x_i \bar{F}_{X_k}(x_k)}{x_1 \bar{F}_{X_1}(x_1)},$$

and

$$\frac{\mathbb{E}[(X_k - x_k)_+]}{x_i \bar{F}_{X_k}(x_k)} \frac{x_i \bar{F}_{X_k}(x_k)}{x_1 \bar{F}_{X_1}(x_1)} \underset{\alpha \rightarrow 1}{\sim} \frac{c_k}{\theta - 1} \left( \frac{x_k}{x_1} \right)^{-\theta+1}.$$

Finally, we deduce that

$$\begin{aligned} - \sum_{k \in J_0^1 \cup J_C^1 \setminus J_\infty^1} \overline{\lim}_{\alpha \rightarrow 1} \frac{x_k}{x_1} &\leq \lim_{\alpha \rightarrow 1} \sum_{k \in J_0^1 \cup J_C^1 \setminus J_\infty^1} \frac{l_{X_k, X_1}^\alpha(x_k, x_1)}{x_1 \bar{F}_{X_1}(x_1)} \\ &\leq \overline{\lim}_{\alpha \rightarrow 1} \sum_{k \in J_0^1 \cup J_C^1 \setminus J_\infty^1} \frac{l_{X_k, X_1}^\alpha(x_k, x_1)}{x_1 \bar{F}_{X_1}(x_1)} \\ &\leq \sum_{k \in J_C^1} \left( \frac{c_k}{\theta - 1} \left( \overline{\lim}_{\alpha \rightarrow 1} \frac{x_k}{x_1} \right)^{-\theta+1} - \ell \lim_{\alpha \rightarrow 1} \frac{x_k}{x_1} \right) + \sum_{k \in J_0^1 \setminus J_\infty^1} \left( 1 + \frac{c_k}{\theta - 1} \right). \end{aligned}$$

This is contradictory with (2.11), and consequently  $J_\infty^1$  is necessarily an empty set. The result follows.  $\square$

**Proposition 2.5** (Asymptotic multivariate expectile). *Assume that H1 and H2 are satisfied and  $\mathbf{X}$  has a regularly varying multivariate distribution in the sense of Definition 1.6, consider the  $L_1$ -expectiles  $\mathbf{e}_\alpha(\mathbf{X}) = (\mathbf{e}_\alpha^i(\mathbf{X}))_{i=1, \dots, d}$ . Then any limit vector  $(\eta, \beta_2, \dots, \beta_d)$  of  $\left( \frac{1-\alpha}{F_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))}, \frac{\mathbf{e}_\alpha^2(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})}, \dots, \frac{\mathbf{e}_\alpha^d(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})} \right)$  satisfies the following equation system*

$$(2.12) \quad \frac{1}{\theta - 1} - \eta \frac{(\beta_k)^\theta}{c_k} = - \sum_{i=1, i \neq k}^d \left( \int_{\frac{\beta_i}{\beta_k}}^{+\infty} \lambda_U^{ik} \left( \frac{c_i}{c_k} t^{-\theta}, 1 \right) dt - \eta \frac{\beta_k^{\theta-1}}{c_k} \beta_i \right), \forall k \in \{1, \dots, d\}.$$

By solving the system (2.12), we may obtain an equivalent of the asymptotic multivariate expectile, using the marginal quantiles.

*Proof.* The optimality system (2.3) can be written in the following form

$$\begin{aligned} (2\alpha - 1) \frac{\mathbb{E}[(X_k - x_k)_+]}{x_k \bar{F}_{X_k}(x_k)} - \frac{1 - \alpha}{\bar{F}_{X_k}(x_k)} \left( 1 - \frac{\mathbb{E}[X_k]}{x_k} \right) &= \sum_{i=1, i \neq k}^d \left( (1 - \alpha) \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_k < x_k\}}]}{x_k \bar{F}_{X_k}(x_k)} \right) \\ &\quad - \sum_{i=1, i \neq k}^d \alpha \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_k > x_k\}}]}{x_k \bar{F}_{X_k}(x_k)}, \quad \forall k \in \{1, \dots, d\}. \end{aligned}$$

For all  $k \in \{1, \dots, d\}$ , we have (taking if necessary a subsequence)

$$\lim_{\alpha \rightarrow 1} (2\alpha - 1) \frac{\mathbb{E}[(X_k - x_k)_+]}{x_k \bar{F}_{X_k}(x_k)} - \frac{1 - \alpha}{\bar{F}_{X_k}(x_k)} \left( 1 - \frac{\mathbb{E}[X_k]}{x_k} \right) = \frac{1}{\theta - 1} - \eta \frac{(\beta_k)^\theta}{c_k},$$

and for all  $i \in \{1, \dots, d\} \setminus \{k\}$

$$\begin{aligned} \lim_{\alpha \rightarrow 1} (1 - \alpha) \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_k < x_k\}}]}{x_k \bar{F}_{X_k}(x_k)} &= \lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_k}(x_k)} \left( \frac{x_i}{x_k} \mathbb{P}(X_i < x_i, X_k < x_k) - \frac{\mathbb{E}[X_i \mathbb{1}_{\{X_i < x_i, X_k < x_k\}}]}{x_k} \right) \\ &= \eta \frac{\beta_k^\theta}{c_k} \frac{\beta_i}{\beta_k} = \eta \frac{\beta_k^{\theta-1}}{c_k} \beta_i. \end{aligned}$$

Moreover,

$$\begin{aligned} \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_k > x_k\}}]}{x_k \bar{F}_{X_k}(x_k)} &= \frac{1}{x_k \bar{F}_{X_k}(x_k)} \int_{x_i}^{+\infty} \mathbb{P}(X_i > t, X_k > x_k) dt \\ &= \int_{\frac{x_i}{x_k}}^{+\infty} \frac{\mathbb{P}(X_i > tx_k, X_k > x_k)}{\bar{F}_{X_k}(x_k)} dt \\ &= \int_{\frac{x_i}{x_k}}^{+\infty} \frac{\mathbb{P}(\bar{F}_{X_i}(X_i) < \bar{F}_{X_i}(tx_k), \bar{F}_{X_k}(X_k) < \bar{F}_{X_k}(x_k))}{\bar{F}_{X_k}(x_k)} dt. \end{aligned}$$

Firstly, we remark that

$$\left| \int_{\frac{\beta_i}{\beta_k}}^{\frac{x_i}{x_k}} \frac{\mathbb{P}(\bar{F}_{X_i}(X_i) < \bar{F}_{X_i}(tx_k), \bar{F}_{X_k}(X_k) < \bar{F}_{X_k}(x_k))}{\bar{F}_{X_k}(x_k)} dt \right| \leq \left| \frac{x_i}{x_k} - \frac{\beta_i}{\beta_k} \right|.$$

Since the functions  $\lambda_U^{ik}$  are assumed to be continuous,

$$(2.13) \quad \lim_{\alpha \rightarrow 1} \frac{\mathbb{P}(\bar{F}_{X_i}(X_i) < \bar{F}_{X_i}(tx_k), \bar{F}_{X_k}(X_k) < \bar{F}_{X_k}(x_k))}{\bar{F}_{X_k}(x_k)} = \lambda_U^{ik} \left( \frac{c_i}{c_k} t^{-\theta}, 1 \right).$$

In order to show that

$$\lim_{\alpha \rightarrow 1} \alpha \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_k > x_k\}}]}{x_k \bar{F}_{X_k}(x_k)} = \int_{\frac{\beta_i}{\beta_k}}^{+\infty} \lambda_U^{ik} \left( \frac{c_i}{c_k} t^{-\theta}, 1 \right) dt,$$

we may use the Lebesgue's Dominated Convergence Theorem with Potter's bounds (1942) (Lemma 1.5) for regularly varying functions.

First of all,

$$\frac{\mathbb{P}(\bar{F}_{X_i}(X_i) < \bar{F}_{X_i}(tx_k), \bar{F}_{X_k}(X_k) < \bar{F}_{X_k}(x_k))}{\bar{F}_{X_k}(x_k)} \leq \min \left\{ 1, \frac{\bar{F}_{X_i}(tx_k)}{\bar{F}_{X_k}(x_k)} \right\},$$

since  $\frac{\bar{F}_{X_i}(tx_k)}{\bar{F}_{X_k}(x_k)} = \frac{\bar{F}_{X_i}(tx_k)}{F_{X_k}(tx_k)} \frac{\bar{F}_{X_k}(tx_k)}{\bar{F}_{X_k}(x_k)}$  and  $\lim_{\alpha \rightarrow 1} \frac{\bar{F}_{X_i}(tx_k)}{F_{X_k}(tx_k)} = \frac{c_i}{c_k}$ , using Potter's bounds, for all  $\varepsilon_1 > 0$  and  $0 < \varepsilon_2 < \theta - 1$ , there exists  $x_k^0(\varepsilon_2, \varepsilon_1)$  such that for  $\min\{x_k, tx_k\} \geq x_k^0(\varepsilon_2, \varepsilon_1)$

$$\frac{\bar{F}_{X_i}(tx_k)}{\bar{F}_{X_k}(x_k)} \leq \left( \frac{c_i}{c_k} + 2\varepsilon_1 \right) t^{-\theta} \max(t^{\varepsilon_2}, t^{-\varepsilon_2}).$$

Lebesgue's theorem gives

$$\lim_{\alpha \rightarrow 1} \int_{\frac{x_i}{x_k}}^{+\infty} \frac{\mathbb{P}(\bar{F}_{X_i}(X_i) < \bar{F}_{X_i}(tx_k), \bar{F}_{X_k}(X_k) < \bar{F}_{X_k}(x_k))}{\bar{F}_{X_k}(x_k)} dt = \int_{\frac{\beta_i}{\beta_k}}^{+\infty} \lambda_U^{ik} \left( \frac{c_i}{c_k} t^{-\theta}, 1 \right) dt,$$

so, for all  $(i \neq k) \in \{1, \dots, d\}^2$

$$\lim_{\alpha \rightarrow 1} \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_k > x_k\}}]}{x_k \bar{F}_{X_k}(x_k)} = \int_{\frac{\beta_i}{\beta_k}}^{+\infty} \lambda_U^{ik} \left( \frac{c_i}{c_k} t^{-\theta}, 1 \right) dt.$$

Hence the system announced in this proposition.  $\square$

In the general case of  $\Sigma$ -expectiles, with  $\Sigma = (\pi_{i,j})_{i,j=1,\dots,d}$ ,  $\pi_{i,j} \geq 0$ ,  $\pi_{ii} = \pi_i > 0$ , System (2.12) becomes

$$\frac{1}{\theta - 1} - \eta \frac{(\beta_k)^\theta}{c_k} = - \sum_{i=1, i \neq k}^d \frac{\pi_{ik}}{\pi_k} \left( \int_{\frac{\beta_i}{\beta_k}}^{+\infty} \lambda_U^{ik} \left( \frac{c_i}{c_k} t^{-\theta}, 1 \right) dt - \eta \frac{\beta_k^{\theta-1}}{c_k} \beta_i \right), \forall k \in \{1, \dots, d\}.$$

Moreover, let us remark that System (2.12) is equivalent to the following system

$$(2.14) \quad \sum_{i=1}^d \int_{\frac{\beta_i}{\beta_k}}^{+\infty} \lambda_U^{ik} (c_i t^{-\theta}, c_k \beta_k^{-\theta}) dt = \sum_{i=1}^d \int_{\beta_i}^{+\infty} \lambda_U^{i1} (c_i t^{-\theta}, 1) dt, \forall k \in \{2, \dots, d\}.$$

The limit points  $\beta_i$  are thus completely determined by the asymptotic bivariate dependencies between the marginal components of the vector  $\mathbf{X}$ .

**Proposition 2.6.** *Assume that H1 and H2 are satisfied and the multivariate distribution of  $\mathbf{X}$  is regularly varying in the sense of Definition 1.6, consider the  $L_1$ -expectiles  $\mathbf{e}_\alpha(\mathbf{X}) = (\mathbf{e}_\alpha^i(\mathbf{X}))_{i=1, \dots, d}$ . Then any limit vector  $(\eta, \beta_2, \dots, \beta_d)$  of  $\left( \frac{1-\alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))}, \frac{\mathbf{e}_\alpha^2(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})}, \dots, \frac{\mathbf{e}_\alpha^d(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})} \right)$  satisfies the following system of equations,  $\forall k \in \{1, \dots, d\}$*

$$(2.15) \quad \frac{1}{\theta-1} - \eta \frac{(\beta_k)^\theta}{c_k} = - \sum_{i=1, i \neq k}^d \left( \frac{c_i}{c_k} \left( \frac{\beta_i}{\beta_k} \right)^{-\theta+1} \int_1^{+\infty} \lambda_U^{ik} \left( t^{-\theta}, \frac{c_k}{c_i} \left( \frac{\beta_k}{\beta_i} \right)^{-\theta} \right) dt - \eta \frac{\beta_k^{\theta-1}}{c_k} \beta_i \right).$$

*Proof.* The proof is straightforward using a substitution in System (2.12) and the positive homogeneity property of the bivariate tail dependence functions  $\lambda_U^{ik}$  (see Proposition 2.2 in [? ]).  $\square$

The main utility of writing the asymptotic optimality system in the form (2.15) is the possibility to give an explicit form to  $(\eta, \beta_2, \dots, \beta_d)$  for some dependence structures.

**Example:** Consider that the dependence structure of  $\mathbf{X}$  is given by an Archimedean copula with generator  $\psi$ . The survival copula is given by

$$\bar{C}(x_1, \dots, x_d) = \psi(\psi^\leftarrow(x_1) + \dots + \psi^\leftarrow(x_d)),$$

where  $\psi^\leftarrow(x) = \inf\{t \geq 0 | \psi(t) \leq x\}$  (see e.g. [?] for more details). Assume that,  $\psi$  is a regularly varying function of a non-positive index  $\psi \in \text{RV}_{-\theta_\psi}$ . According to [?], the right tail dependence functions exist, and one can get their explicit forms

$$\lambda_U^k(x_1, \dots, x_k) = \left( \sum_{i=1}^k x_i^{-\frac{1}{\theta_\psi}} \right)^{-\theta_\psi}.$$

Thus, the bivariate upper tail dependence functions are given by

$$\lambda_U^{ik} \left( t^{-\theta}, \frac{c_k}{c_i} \left( \frac{\beta_k}{\beta_i} \right)^{-\theta} \right) = \left( t^{\frac{\theta}{\theta_\psi}} + \left( \frac{c_i}{c_k} \right)^{\frac{1}{\theta_\psi}} \left( \frac{\beta_k}{\beta_i} \right)^{\frac{\theta}{\theta_\psi}} \right)^{-\theta_\psi}.$$

In particular, if  $\theta = \theta_\psi$ , we have

$$\int_1^{+\infty} \lambda_U^{ik} \left( t^{-\theta}, \frac{c_k}{c_i} \left( \frac{\beta_k}{\beta_i} \right)^{-\theta} \right) dt = \frac{1}{\theta-1} \left( 1 + \left( \frac{c_i}{c_k} \right)^{\frac{1}{\theta}} \frac{\beta_k}{\beta_i} \right)^{-\theta+1},$$

and System 2.15 becomes

$$\frac{1}{\theta-1} - \eta \frac{(\beta_k)^\theta}{c_k} = - \sum_{i=1, i \neq k}^d \left( \frac{1}{\theta-1} \frac{c_i}{c_k} \left( \frac{\beta_i}{\beta_k} + \left( \frac{c_i}{c_k} \right)^{\frac{1}{\theta}} \right)^{-\theta+1} - \eta \frac{\beta_k^{\theta-1}}{c_k} \beta_i \right).$$

$\square$

**Lemma 2.7** (The comonotonic Fréchet case). *Under H1 and H2, consider the  $L_1$ -expectiles  $\mathbf{e}_\alpha(\mathbf{X}) = (\mathbf{e}_\alpha^i(\mathbf{X}))_{i=1, \dots, d}$ . If  $\mathbf{X} = (X_1, \dots, X_d)$  is a comonotonic random vector, then the limit*

$$(\eta, \beta_2, \dots, \beta_d) = \lim_{\alpha \rightarrow 1} \left( \frac{1-\alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))}, \frac{\mathbf{e}_\alpha^2(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})}, \dots, \frac{\mathbf{e}_\alpha^d(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})} \right),$$

satisfies

$$\lim_{\alpha \rightarrow 1} \frac{1-\alpha}{\bar{F}_{X_k}(\mathbf{e}_\alpha^k(\mathbf{X}))} = \frac{1}{\theta-1} \quad \text{and} \quad \beta_k = c_k^{1/\theta}, \quad \forall k \in \{1, \dots, d\}.$$

*Proof.* Since the random vector  $\mathbf{X}$  is comonotonic, its survival copula is

$$\overline{C}_{\mathbf{X}}(u_1, \dots, u_d) = \min(u_1, \dots, u_d), \quad \forall (u_1, \dots, u_d) \in [0, 1]^d.$$

We deduce the expression of the functions  $\lambda_U^{ij}$

$$\lambda_U^{ij}(x_i, x_j) = \min(x_i, x_j), \quad \forall (x_i, x_j) \in \mathbb{R}_+^2, \forall i, j \in \{1, \dots, d\}.$$

So,

$$\begin{aligned} \int_1^{+\infty} \lambda_U^{ik} \left( t^{-\theta}, \frac{c_k}{c_i} \left( \frac{\beta_k}{\beta_i} \right)^{-\theta} \right) dt &= \int_1^{+\infty} \min \left( t^{-\theta}, \frac{c_k}{c_i} \left( \frac{\beta_k}{\beta_i} \right)^{-\theta} \right) dt \\ &= \left( \frac{\beta_k}{\beta_i} \left( \frac{c_k}{c_i} \right)^{-\frac{1}{\theta}} - 1 \right)_+ \frac{c_k}{c_i} \left( \frac{\beta_k}{\beta_i} \right)^{-\theta} \\ &\quad + \frac{1}{\theta - 1} \left( 1 + \left( \frac{\beta_k}{\beta_i} \left( \frac{c_k}{c_i} \right)^{-\frac{1}{\theta}} - 1 \right)_+ \right)^{-\theta+1}. \end{aligned}$$

Under assumptions H1 and H2, and by Proposition 2.15, let  $(\eta, \beta_2, \dots, \beta_d)$  be a solution of the following equation system.

$$\begin{aligned} \eta \sum_{i=1}^d \beta_i - \frac{1}{\theta - 1} \sum_{i=1}^d c_i \beta_i^{-\theta+1} &= \sum_{i=1, i \neq k}^d c_k \beta_k^{-\theta} \beta_i \left( \frac{\beta_k}{\beta_i} \left( \frac{c_k}{c_i} \right)^{-\frac{1}{\theta}} - 1 \right)_+ \\ &\quad + \frac{1}{\theta - 1} \sum_{i=1, i \neq k}^d c_i \beta_i^{-\theta+1} \left[ \left( 1 + \left( \frac{\beta_k}{\beta_i} \left( \frac{c_k}{c_i} \right)^{-\frac{1}{\theta}} - 1 \right)_+ \right)^{-\theta+1} - 1 \right], \end{aligned}$$

$\forall k \in \{1, \dots, d\}$ .  $\eta = \frac{1}{\theta-1}$  and  $\beta_k = c_k^{\frac{1}{\theta-1}}$  is the only solution to this system.  $\square$

**Proposition 2.8** (Asymptotic independence case). *Under H1 and H2, consider the  $L_1$ -expectiles  $\mathbf{e}_\alpha(\mathbf{X}) = (\mathbf{e}_\alpha^i(\mathbf{X}))_{i=1, \dots, d}$ . If  $\mathbf{X} = (X_1, \dots, X_d)$  is such that the pairs  $(X_i, X_j)$  are asymptotically independent, then the limit vector  $(\eta, \beta_2, \dots, \beta_d)$  of  $\left( \frac{1-\alpha}{F_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))}, \frac{\mathbf{e}_\alpha^2(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})}, \dots, \frac{\mathbf{e}_\alpha^d(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})} \right)$  satisfies*

$$\eta = \frac{1}{(\theta - 1) \left( 1 + \sum_{j=2}^d c_j^{\frac{1}{\theta-1}} \right)} \text{ and } \beta_k = c_k^{\frac{1}{\theta-1}},$$

for all  $k \in \{1, \dots, d\}$ .

*Proof.* The hypothesis of asymptotic bivariate independence means:

$$\lim_{\alpha \rightarrow 1} \frac{\mathbb{P}(X_i > x_i, X_j > x_j)}{\mathbb{P}(X_j > x_j)} = \lim_{\alpha \rightarrow 1} \frac{\mathbb{P}(X_i > tx_j, X_j > x_j)}{\mathbb{P}(X_j > x_j)} = 0,$$

for all  $(i, j) \in \{1, \dots, d\}^2$  and for all  $t > 0$ , then, Lebesgue's Theorem used as in Proposition 2.5 gives

$$\begin{aligned} \lim_{\alpha \rightarrow 1} \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_j > x_j\}}]}{x_j \bar{F}_{X_j}(x_j)} &= \lim_{\alpha \rightarrow 1} \int_{\frac{x_i}{x_j}}^{+\infty} \frac{\mathbb{P}(X_i > tx_j, X_j > x_j)}{\mathbb{P}(X_j > x_j)} dt \\ &= 0. \end{aligned}$$

The asymptotic multivariate expectile verifies the following equation system

$$\frac{1}{\theta - 1} - \frac{\eta}{c_k} \beta_k^\theta = + \sum_{i=1, i \neq k}^d \frac{\eta}{c_k} \beta_k^{\theta-1} \beta_i, \quad \forall k \in \{1, \dots, d\},$$

which can be rewritten as

$$(2.16) \quad \frac{c_k}{\eta(\theta - 1)\beta_k^{\theta-1}} = \sum_{i=1}^d \beta_i, \quad \forall k \in \{1, \dots, d\},$$

hence  $\beta_k = c_k^{\frac{1}{\theta-1}}$  for all  $k \in \{1, \dots, d\}$ , and

$$\eta = \frac{1}{(\theta - 1) \left( 1 + \sum_{j=2}^d c_j^{\frac{1}{\theta-1}} \right)}.$$

□

In the general case of a matrix of positive coefficients  $\pi_{ij}$ ,  $i, j \in \{1, \dots, d\}$ , the limits  $\beta_i, i = 2, \dots, d$  remain the same, but the limit  $\eta$  will change:

$$\lim_{\alpha \rightarrow 1} \frac{\mathbf{e}_\alpha^k(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})} = c_k^{\frac{1}{\theta-1}} \text{ and } \lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_k}(\mathbf{e}_\alpha^k(\mathbf{X}))} = \frac{c_k^{\frac{1}{\theta-1}}}{(\theta - 1) \left( 1 + \sum_{j=2}^d \frac{\pi_{jk}}{\pi_k} c_j^{\frac{1}{\theta-1}} \right)},$$

for all  $k \in \{1, \dots, d\}$ .

We remark that

$$\lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_i}(x_i)} \leq \frac{c_i^{\frac{1}{\theta-1}}}{\theta - 1},$$

which allows a comparison between the marginal quantile and the corresponding component of the multivariate expectile, and since  $F_{X_k}^{-1}(1 - \cdot)$  is regularly varying function at 0 for all  $k \in \{1, \dots, d\}$  with index  $-\frac{1}{\theta}$  (see Lemma 1.3), we get

$$e_\alpha^k(\mathbf{X}) \underset{\alpha \rightarrow 1}{\sim} \text{VaR}_\alpha(X_k) (\theta - 1)^{-\frac{1}{\theta}} \left( \frac{1 + \sum_{i=2}^d c_i^{\frac{1}{\theta-1}}}{c_k^{\frac{1}{\theta-1}}} \right)^{-\frac{1}{\theta}},$$

where  $\text{VaR}_\alpha(X_k)$  denotes the Value at Risk of  $X_k$  at level  $\alpha$ , ie the  $\alpha$ -quantile  $F_{X_k}^\leftarrow(\alpha)$  of  $X_k$ . These conclusions coincide with the results obtained in dimension 1, for distributions that belong to the domain of attraction of Fréchet, in [? ]. The values of constants  $c_i$  determine the position of the marginal quantile compared to the corresponding component of the multivariate expectile for each risk.

### 3. FRÉCHET MODEL WITH A DOMINANT TAIL

This section is devoted to the case where  $X_1$  has a dominant tail with respect to the  $X_i$ 's. We shall not give the proofs, they are an adaptation to the proofs of Section 2. We shall only state the intermediate needed results and leave the proofs to the reader.

**Proposition 3.1** (Asymptotic dominance). *Under H1, consider the  $L_1$ -expectiles  $\mathbf{e}_\alpha(\mathbf{X}) = (\mathbf{e}_\alpha^i(\mathbf{X}))_{i=1, \dots, d}$ . If*

$$\lim_{x \uparrow +\infty} \frac{\bar{F}_{X_i}(x)}{\bar{F}_{X_1}(x)} = 0, \quad \forall i \in \{2, \dots, d\}, \text{ (dominant tail hypothesis)}$$

then

$$\beta_i = \lim_{\alpha \uparrow 1} \frac{\mathbf{e}_\alpha^i(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})} = 0, \quad \lim_{\alpha \uparrow 1} \frac{1 - \alpha}{\bar{F}_{X_i}(\mathbf{e}_\alpha^i(\mathbf{X}))} = 0, \quad \forall i \in \{2, \dots, d\},$$

and

$$\lim_{\alpha \uparrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))} = \frac{1}{\theta - 1}.$$

The proof of Proposition 3.1 follows from the following three lemmas.

**Lemma 3.2.** *Under H1, consider the  $L_1$ -expectiles  $\mathbf{e}_\alpha(\mathbf{X}) = (\mathbf{e}_\alpha^i(\mathbf{X}))_{i=1, \dots, d}$ . If*

$$\lim_{x \uparrow +\infty} \frac{\bar{F}_{X_i}(x)}{\bar{F}_{X_1}(x)} = 0, \quad \forall i \in \{2, \dots, d\},$$

then

$$\lim_{\alpha \uparrow 1} \frac{\mathbb{E}[(X_i - x_i)_+ \mathbb{1}_{\{X_1 > x_1\}}]}{x_1 \bar{F}_{X_1}(x_1)} = 0, \quad \forall i \in \{2, \dots, d\}.$$

**Lemma 3.3.** Under H1, consider the  $L_1$ -expectiles  $\mathbf{e}_\alpha(\mathbf{X}) = (\mathbf{e}_\alpha^i(\mathbf{X}))_{i=1,\dots,d}$ . If

$$\lim_{x \uparrow +\infty} \frac{\bar{F}_{X_i}(x)}{\bar{F}_{X_1}(x)} = 0, \quad \forall i \in \{2, \dots, d\},$$

then

$$\lim_{\alpha \uparrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))} < +\infty.$$

**Lemma 3.4.** Under H1, consider the  $L_1$ -expectiles  $\mathbf{e}_\alpha(\mathbf{X}) = (\mathbf{e}_\alpha^i(\mathbf{X}))_{i=1,\dots,d}$ . If

$$\lim_{x \uparrow +\infty} \frac{\bar{F}_{X_i}(x)}{\bar{F}_{X_1}(x)} = 0, \quad \forall i \in \{2, \dots, d\},$$

then

$$0 < \lim_{\alpha \uparrow 1} \frac{1 - \alpha}{\bar{F}_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))}.$$

Proposition 3.1 shows that the dominant risk behaves asymptotically as in the univariate case, and its component in the asymptotic multivariate expectile satisfies

$$\mathbf{e}_\alpha^1(\mathbf{X}) \stackrel{\alpha \uparrow 1}{\sim} (\theta - 1)^{-\frac{1}{\theta}} \text{VaR}_\alpha(X_1) \stackrel{\alpha \uparrow 1}{\sim} \mathbf{e}_\alpha(X_1),$$

the right equivalence is proved, in the univariate case, in [? ], Proposition 2.3.

**Example:** Consider Pareto distributions,  $X_i \sim Pa(a_i, b)$ ,  $i = 1, \dots, d$ , such that  $a_i > a_1$  for all  $i \in \{1, \dots, d\}$ . The tail of  $X_1$  dominates that of the  $X_i$ 's and Proposition 3.1 applies.

#### 4. ESTIMATION OF THE ASYMPTOTIC EXPECTILES

In this section, we propose some estimators of the asymptotic multivariate expectile. We focus on the cases of asymptotic independence and comonotonicity, for which the equation system is more tractable. We begin with the main ideas of our approach, then, we construct the estimators using the extreme values statistical tools and prove its consistency. We terminate this section with a simulation study.

**Proposition 4.1** (Estimation's idea). *Using notations of previous sections, consider the  $L_1$ -expectiles  $\mathbf{e}_\alpha(\mathbf{X}) = (\mathbf{e}_\alpha^i(\mathbf{X}))_{i=1,\dots,d}$ . Under H1, H2 and the assumption that the vector  $\left( \frac{1-\alpha}{F_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))}, \frac{\mathbf{e}_\alpha^2(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})}, \dots, \frac{\mathbf{e}_\alpha^d(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})} \right)$  has a unique limit point  $(\eta, \beta_2, \dots, \beta_d)$ ,*

$$\mathbf{e}_\alpha(\mathbf{X})^T \underset{\alpha \rightarrow 1}{\sim} \text{VaR}_\alpha(X_1) \eta^{\frac{1}{\theta}} (1, \beta_2, \dots, \beta_d)^T.$$

*Proof.* Let  $(\eta, \beta_2, \dots, \beta_d) = \lim_{\alpha \rightarrow 1} \left( \frac{1-\alpha}{F_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))}, \frac{\mathbf{e}_\alpha^2(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})}, \dots, \frac{\mathbf{e}_\alpha^d(\mathbf{X})}{\mathbf{e}_\alpha^1(\mathbf{X})} \right)$ , we have

$$\mathbf{e}_\alpha(\mathbf{X})^T \underset{\alpha \rightarrow 1}{\sim} \mathbf{e}_\alpha^1(\mathbf{X}) (1, \beta_2, \dots, \beta_d)^T.$$

Moreover,  $\lim_{\alpha \rightarrow 1} \frac{1-\alpha}{F_{X_1}(\mathbf{e}_\alpha^1(\mathbf{X}))} = \eta$ , and Theorem 1.5.12 in [? ] states that  $F_{X_1}^{\leftarrow}(1 - \cdot)$  is regularly varying at 0, with index  $-\frac{1}{\theta}$ . This leads to

$$\mathbf{e}_\alpha^1(\mathbf{X}) \underset{\alpha \rightarrow 1}{\sim} F_{X_1}^{\leftarrow}(\alpha) \left( \frac{1}{\eta} \right)^{-\frac{1}{\theta}},$$

and the result follows.  $\square$

Proposition 4.1 gives a way to estimate the asymptotic multivariate expectile. Let  $\mathbf{X} = (X_1, \dots, X_d)^T$  be an independent sample of size  $n$  of  $\mathbf{X}$ , with  $\mathbf{X}_i = (X_{1,i}, \dots, X_{d,i})^T$  for all  $i \in \{1, \dots, n\}$ . We denote by  $X_{i,1,n} \leq X_{i,2,n} \leq \dots \leq X_{i,n,n}$  the ordered sample corresponding to  $X_i$ .

**4.1. Estimator's construction.** We begin with the case of asymptotic independence. Propositions 2.8 and 4.1 are the key tools in the construction of the estimator. We have for all  $i \in \{1, \dots, d\}$

$$\beta_i = c_i^{\frac{1}{\theta-1}}, \text{ and } \lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_i}(\mathbf{e}_\alpha^i(\mathbf{X}))} = \frac{c_i^{\frac{1}{\theta-1}}}{(\theta - 1) \left( 1 + \sum_{j=2}^d c_j^{\frac{1}{\theta-1}} \right)}.$$

Proposition 4.1 gives

$$\mathbf{e}_\alpha(\mathbf{X}) \stackrel{\alpha \uparrow 1}{\sim} \text{VaR}_\alpha(X_1) (\theta - 1)^{-\frac{1}{\theta}} \left( 1 + \sum_{i=2}^d c_i^{\frac{1}{\theta-1}} \right)^{-\frac{1}{\theta}} \left( 1, c_2^{\frac{1}{\theta-1}}, \dots, c_d^{\frac{1}{\theta-1}} \right)^T.$$

So, in order to estimate the asymptotic multivariate expectile, we need an estimator of the univariate quantile of  $X_1$ , of the tail equivalence parameters. and of  $\theta$ .

In the same way, and for the case of comonotonic risks, we may use Proposition 2.7

$$\lim_{\alpha \rightarrow 1} \frac{1 - \alpha}{\bar{F}_{X_i}(\mathbf{e}_\alpha^i(\mathbf{X}))} = \frac{1}{\theta - 1} \quad \text{and} \quad \beta_i = c_i^{1/\theta}, \quad \forall i \in \{1, \dots, d\},$$

and by Proposition 4.1 we obtain

$$\mathbf{e}_\alpha(\mathbf{X})^T \stackrel{\alpha \uparrow 1}{\sim} \text{VaR}_\alpha(X_1) (\theta - 1)^{-\frac{1}{\theta}} (1, c_2^{\frac{1}{\theta}}, \dots, c_d^{\frac{1}{\theta}})^T.$$

The  $X_i$ 's have all the same index  $\theta$  of regular variation, which is also the same as the index of regular variation of  $\|\mathbf{X}\|$ . We propose to estimate  $\theta$  by using the Hill estimator  $\hat{\gamma}$ . We shall denote  $\hat{\theta} = \frac{1}{\hat{\gamma}}$ . See [?] for details on the Hill estimator. In order to estimate the  $c_i$ 's, we shall use the GPD approximation: for  $u$  a large threshold, and  $x \geq u$ ,

$$\bar{F}(x) \sim \bar{F}(u) \left( \frac{x}{u} \right)^{-\theta},$$

Let  $k \in \mathbb{N}$  be fixed and consider the thresholds  $u_i$ :

$$\bar{F}_{X_i}(u_i) = \bar{F}_{X_1}(u_1) = \frac{k}{n}, \quad \forall i \in \{1, \dots, d\},$$

The  $u_i$  are estimated by  $X_{i,n-k+1,n}$  with  $k \rightarrow \infty$  and  $k/n \rightarrow 0$  as  $n \rightarrow \infty$ . Using Lemma 2.2, we get

$$c_i = \lim_{n \rightarrow \infty} \left( \frac{u_i}{u_1} \right)^\theta.$$

We shall consider

$$(4.1) \quad \hat{c}_i = \left( \frac{X_{i,n-k+1,n}}{X_{1,n-k+1,n}} \right)^{\frac{1}{\hat{\gamma}(k)}},$$

where  $\hat{\gamma}(k)$  is the Hill's estimator of the extreme values index constructed using the  $k$  largest observations of  $\|\mathbf{X}\|$ . Let  $\hat{\theta} = \frac{1}{\hat{\gamma}(k)}$ .

**Proposition 4.2.** *Let  $k = k(n)$  be such that  $k \rightarrow \infty$  and  $k/n \rightarrow 0$  as  $n \rightarrow \infty$ . Under H1 and H2, for any  $i = 2, \dots, d$ ,*

$$\hat{c}_i \xrightarrow{\mathbb{P}} c_i.$$

*Proof.* The results in [?] page 86 imply that for any  $i = 1, \dots, d$

$$\frac{X_{i,n-k+1,n}}{u_i} \xrightarrow{\mathbb{P}} 1.$$

Moreover, this is well known (see [?]) that the Hill estimator is consistent. Using (4.1), and the fact that

$$\frac{X_{i,n-k+1,n}}{X_{1,n-k+1,n}} \sim \frac{u_i}{u_1} \text{ in probability and thus is bounded in probability,}$$

we get the result. □

To estimate the extreme quantile, we may also use Weissman's estimator (1978) [?]:

$$\widehat{\text{VaR}}_\alpha(X_1) = X_{1,n-k(n)+1,n} \left( \frac{k(n)}{(1-\alpha)n} \right)^{\hat{\gamma}}.$$

The properties of Weissman's estimator are presented in Embrechts et al. (1997) [?] and also in [?] page 119. Now, we can deduce some estimators of the extreme multivariate expectile, using the previous ones, in the cases of asymptotic independence and perfect dependence.

**Definition 4.3** (Multivariate expectile estimator, Asymptotic independence). Under H1 and H2, in the case of bivariate asymptotic independence of the random vector  $\mathbf{X} = (X_1, \dots, X_d)^T$ , we define the estimator of the  $L_1$ -expectile as follows

$$\hat{\mathbf{e}}_\alpha^\perp(\mathbf{X}) = X_{1, n-k(n)+1, n} \left( \frac{k(n)}{(1-\alpha)n} \right)^{\hat{\gamma}} \left( \frac{\hat{\gamma}}{1-\hat{\gamma}} \right)^{\hat{\gamma}} \left( \frac{1}{1 + \sum_{\ell=2}^d \hat{c}_\ell^{\frac{\hat{\gamma}}{1-\hat{\gamma}}}} \right)^{\hat{\gamma}} \left( 1, \hat{c}_2^{\frac{\hat{\gamma}}{1-\hat{\gamma}}}, \dots, \hat{c}_d^{\frac{\hat{\gamma}}{1-\hat{\gamma}}} \right)^T.$$

Propositions 2.8 and 4.2 as well as convergence properties of  $\widehat{\text{VaR}}_\alpha(X_1)$  imply that the term by term ratio  $\hat{\mathbf{e}}_\alpha^\perp(\mathbf{X})/\mathbf{e}_\alpha(\mathbf{X})$  goes to 1 in probability. More work is required to get the asymptotic normality.

**Definition 4.4** (Multivariate expectile estimator, comonotonic risks). Under the assumptions of the Fréchet model with equivalent tails, for a comonotonic random vector  $\mathbf{X} = (X_1, \dots, X_d)^T$ , we define the estimator of  $L_1$ -expectile as follows

$$\hat{\mathbf{e}}_\alpha^+(\mathbf{X}) = X_{1, n-k(n)+1, n} \left( \frac{k(n)}{(1-\alpha)n} \right)^{\hat{\gamma}} \left( \frac{\hat{\gamma}}{1-\hat{\gamma}} \right)^{\hat{\gamma}} \left( 1, \hat{c}_2^{\hat{\gamma}}, \dots, \hat{c}_d^{\hat{\gamma}} \right)^T.$$

As above, Propositions 2.8 and 4.2 as well as convergence properties of  $\widehat{\text{VaR}}_\alpha(X_1)$  imply that the term by term ratio  $\hat{\mathbf{e}}_\alpha^+(\mathbf{X})/\mathbf{e}_\alpha(\mathbf{X})$  goes to 1 in probability.

**4.2. Numerical illustration.** The attraction domain of Fréchet contains the usual distributions of Pareto, Student and Cauchy. In order to illustrate the convergence of the proposed estimators, we consider a bivariate Pareto model  $X_i \sim Pa(a, b_i)$ ,  $i \in \{1, 2\}$ . Both distributions have the same scale parameter  $a$ , so they have equivalent tails with equivalence parameter

$$c_2 = \lim_{x \rightarrow +\infty} \frac{\bar{F}_{X_2}(x)}{\bar{F}_{X_1}(x)} = \lim_{x \rightarrow +\infty} \frac{\left( \frac{b_2}{b_2+x} \right)^a}{\left( \frac{b_1}{b_1+x} \right)^a} = \left( \frac{b_2}{b_1} \right)^a.$$

In what follows, we consider two models for which the exact values of the  $L_1$ -expectiles are computable. In the first model, the  $X_i$ 's are independent. In the second one, the  $X_i$ 's are comonotonic. In the simulations below, we have taken the same  $k = k(n)$  to get  $\hat{\gamma}$  and  $\hat{\mathbf{e}}_\alpha(\mathbf{X})$ . For  $X_i \sim Pa(2, 5 \cdot (i+1))_{i=1,2}$ , and  $n = 100000$ , Figure 1 illustrates the convergence of estimator  $\hat{c}_2$  when  $k \rightarrow +\infty$ .

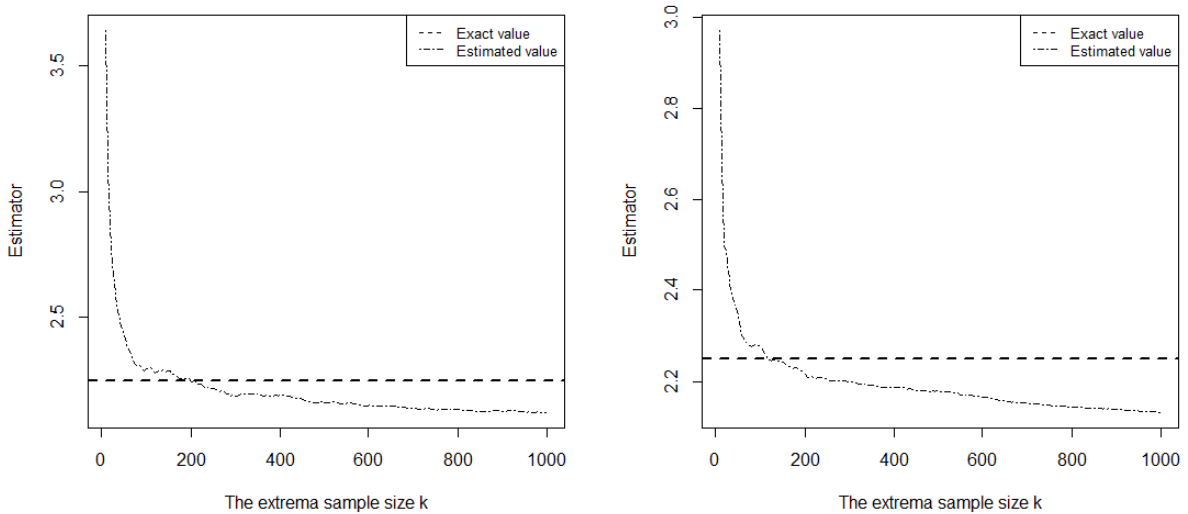


FIGURE 1. Convergence of  $\hat{c}_2$ .  $X_i \sim Pa(2, 5 \cdot (i+1))_{i=1,2}$ . The left figure concerns the independence model. The right one concerns the comonotonicity case.

In the independence case which is a special case of asymptotic independence, the functions  $l_{X_i, X_j}^\alpha$  defined in (2.2) have the following expression

$$l_{X_i, X_j}^\alpha(x_i, x_j) = \alpha \left( \bar{F}_{X_j}(x_j) \mathbb{E}[(X_i - x_i)_+] \right) - (1 - \alpha) \left( F_{X_j}(x_j) \mathbb{E}[(X_i - x_i)_-] \right).$$



In the comonotonic case we have

$$l_{X_i, X_j}^\alpha(x_i, x_j) = \alpha \left( \bar{F}_{X_j}(x_j)(\mu_{i,j} - x_i)_+ + \mathbb{E} \left[ (X_i - \max(x_i, \mu_{i,j}))_+ \right] \right) \\ - (1 - \alpha) \left( F_{X_j}(x_j)(x_i - \mu_{i,j})_+ + \mathbb{E} \left[ (X_i - \min(x_i, \mu_{i,j}))_- \right] \right),$$

where  $\mu_{i,j} = F_{X_i}^{\leftarrow}(F_{X_j}(x_j))$ . For Pareto distributions  $\mu_{i,j} = \frac{b_i}{b_j} x_j$ .

From these expressions, the exact value of the asymptotic multivariate expectile is obtained using numerical optimization, and we can confront it to the estimated values. Figure 2 presents the obtained results for different  $k(n)$  values in the independence case, the comonotonic case is illustrated in Figure 3. The simulations parameters are  $a = 2$ ,  $b_1 = 10$ ,  $b_2 = 15$ , and  $n = 100000$ .

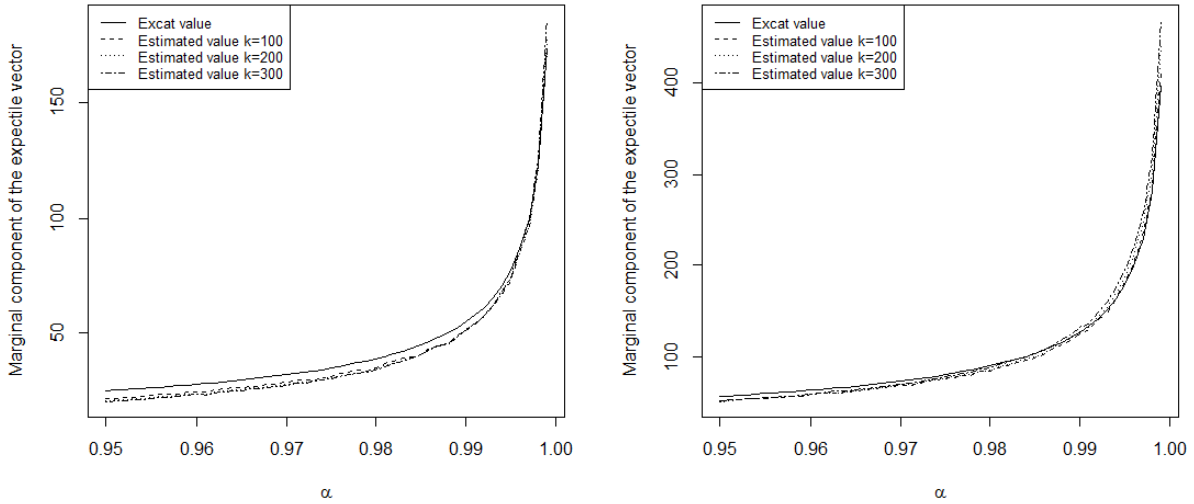


FIGURE 2. Convergence of  $\hat{\mathbf{e}}_\alpha^\perp(\mathbf{X})$  (asymptotic independence case). On the left, the first coordinate of  $\mathbf{e}_\alpha(\mathbf{X})$  and  $\hat{\mathbf{e}}_\alpha^\perp(\mathbf{X})$  for various values of  $k = k(n)$  are plotted. The right figure concerns the second coordinate.

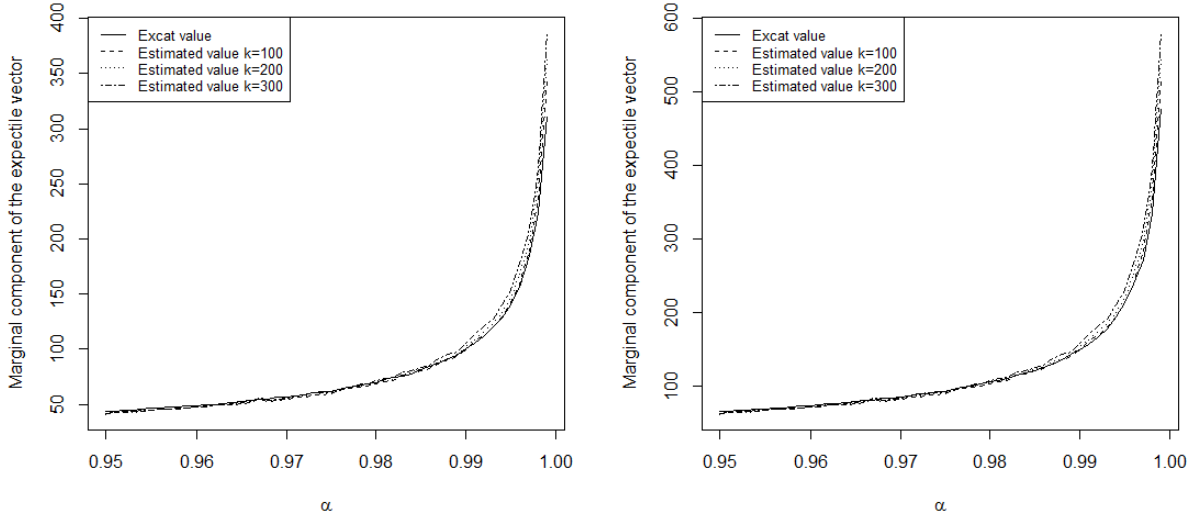


FIGURE 3. Convergence of  $\hat{\mathbf{e}}_{\alpha}^{+}(\mathbf{X})$  (comonotonic case). On the left, the first coordinate of  $\mathbf{e}_{\alpha}(\mathbf{X})$  and  $\hat{\mathbf{e}}_{\alpha}^{+}(\mathbf{X})$  for various values of  $k = k(n)$  are plotted. The right figure concerns the second coordinate.

### CONCLUSION

We have studied properties of extreme expectiles in a regular variations framework. We have seen shows that the asymptotic behavior of expectiles vectors strongly depends on the marginal tails behavior and on the nature of the asymptotic dependence. The main conclusion of this analysis, is that the equivalence of marginal tails leads to equivalence of the asymptotic expectile components.

The statistical estimation of the integrals of the tail dependence functions would allow to construct estimators of the asymptotic expectile vectors. This paper's estimations are limited to the cases of asymptotic independence and comonotonicity which do not require the estimation of the tail dependence functions. The asymptotic normality of the estimators proposed in the last section of this paper requires a careful technical analysis which is not considered in this paper.

A natural perspective of this work, is to study the asymptotic behavior of  $\Sigma$ -expectiles in the case of equivalent tails of marginal distributions in the domains of attraction of Weibull and Gumbel. The Gumbel's domain contains most of the usual distributions, especially the family of Weibull tail-distributions, which makes the analysis of its case an interesting task.

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